Estimation of Error Constants Appearing in Non-conforming Linear Triangular Finite Element

Xuefeng Liu[1*], Fumio Kikuchi[1]

[1] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo, 153-8914, Japan*
e-mail: kikuchi@ms.u-tokyo.ac.jp, xfliu@ms.u-tokyo.ac.jp

Keywords : FEM, non-conforming linear triangle, a priori and a posteriori error estimates, error constants, Raviart-Thomas element

Abstract

As a well-known alternative to the conforming linear triangular finite element for approximation of the first-order Sobolev space, the non-conforming linear element is considered a classical discontinuous Galerkin finite element and has various interesting and attractive properties from both theoretical and practical standpoints. In particular, its a priori error analysis was performed in fairly early stage of mathematical analysis of FEM, and recently a posteriori error analysis is rapidly developing as well. For accurate error estimation of such an FEM, various error constants must be evaluated quantitatively. Based on our preceding works on the constant and conforming linear triangles, we here give some results for error constants required for analysis of the non-conforming linear triangle.
More specifically, we first summarize a priori error estimation of the present non-conforming FEM, where several error constants appear. In this process, we use the lowest-order Raviart-Thomas triangular element to deal with the inter-element discontinuity of the approximate functions. Then we introduce some constants related to a reference triangle, some of which are popular in the constant and conforming linear cases. We give some theoretical results for the upper bounds of such constants. In some very special cases, exact values of constants can be obtained. In particular, a kind of maximum angle condition is required as in the case of the conforming linear triangle. Finally, we illustrate some numerical results to support the validity of such upper bounds. Our results can be effectively used in the quantitative a priori and a posteriori error estimates for the non-conforming linear triangular FEM.

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[1] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro, Tokyo, 153-8914, Japan*

e-mail: xfliu@ms.u-tokyo.ac.jp, kikuchi@ms.u-tokyo.ac.jp

Abstract The non-conforming linear (P_1) triangular FEM can be viewed as a kind of the discontinuous Galerkin method, and is attractive in both theoretical and practical senses. Since various error constants must be quantitatively evaluated for its accurate a priori and a posteriori error estimates, we derive their theoretical upper bounds and some computational results. In particular, the Babuška-Aziz maximum angle condition is required just as in the case of the conforming P_1 triangle. Some applications and numerical results are also illustrated to see the validity and effectiveness of our analysis.

Key Words: FEM, non-conforming linear triangle, a priori and a posteriori error estimates, error constants, Raviart-Thomas element

INTRODUCTION

As a well-known alternative to the conforming linear (P_1) triangular finite element for approximation of the first-order Sobolev space (H^1), the non-conforming P_1 element is considered a classical discontinuous Galerkin finite element [4] and has various interesting properties from both theoretical and practical standpoints [10, 22]. In particular, its a priori error analysis was performed in fairly early stage of mathematical analysis of FEM (Finite Element Method), and recently a posteriori error analysis is rapidly developing as well. For accurate error estimation of such an FEM, various error constants must be evaluated quantitatively [2, 6, 8, 17, 20, 21].

Based on our preceding works on the constant (P_0) and the conforming P_1 triangles [14, 15], we here give some results for error constants required for analysis of the non-conforming P_1 triangle. More specifically, we first summarize a priori error estimation of the present non-conforming FEM, where several error constants appear. In this process, we use the lowest-order Raviart-Thomas triangular $H(\mathrm{div})$ element to deal with the inter-element discontinuity of the approximate functions [9, 16]. Then we introduce some constants related to a reference triangle, some of which are popular in the P_0 and the conforming P_1 cases. We give some theoretical results for the upper bounds of such constants. Finally, we illustrate some numerical results to support the validity of such upper bounds. Our results can be effectively used in the quantitative a priori and a posteriori error estimates for the non-conforming P_1 triangular FEM.

A PRIORI ERROR ESTIMATION

We here summarize a priori error estimation of the non-conforming P_1 triangular FEM. Let Ω be a bounded convex polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$, and let us consider a weak formulation of the Dirichlet boundary value problem for the Poisson equation: *Given* $f \in L_2(\Omega)$, *find* $u \in H_0^1(\Omega)$ *s. t.*

$$(\nabla u, \nabla v) = (f, v)\,; \;\; \forall\, v \in H_0^1(\Omega)\,. \tag{1}$$

Here, $L_2(\Omega)$ and $H_0^1(\Omega)$ are the usual Hilbertian Sobolev spaces associated to Ω, ∇ is the gradient operator, and $(\cdot,\cdot)$ stands for the inner products of both $L_2(\Omega)$ and $L_2(\Omega)^2$. It is well known that the solution exists uniquely in $H_0^1(\Omega)$ and also belongs to $H^2(\Omega)$ for the considered Ω.

Let us consider a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω, to which we associate the non-conforming P_1 finite element spaces $\{V^h\}_{h>0}$. Each V^h is constructed over a certain $\mathcal{T}^h$, and the functions in V^h are linear in each $K \in \mathcal{T}^h$ with continuity only at midpoints of edges, and also vanish at the midpoints on $\partial\Omega$ to approximate the homogeneous Dirichlet condition [10, 22]. Then the finite element solution $u_h \in V^h$ is determined by, for a given $f \in L_2(\Omega)$,

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h)\,; \;\; \forall\, v_h \in V^h\,, \tag{2}$$

where ∇_h is the "non-conforming" or discrete gradient defined as the $L_2(\Omega)^2$-valued operator by the element-wise relations $(\nabla_h v)|K = \nabla(v|K)$ for $\forall v \in V^h + H^1(\Omega)$ and $\forall K \in \mathcal{T}^h$.
Eq. (2) is formally of the same form as in the conforming case, so that, for error analysis, it is natural to consider an appropriate interpolation operator Π_h from $H_0^1(\Omega)$ (or its intersection with some other spaces) to V^h. However, the situation is not so simple. That is, using the Green formula, we have

$$(\nabla_h u_h, \nabla_h v_h) = (\nabla u, \nabla_h v_h) - \sum_{K\in\mathcal{T}^h}\int_{\partial K} v_h \left.\frac{\partial u}{\partial n}\right|_{\partial K} d\gamma\,; \;\; \forall\, v_h \in V^h\,, \tag{3}$$

where $\left.\frac{\partial u}{\partial n}\right|_{\partial K}$ denotes the trace of the derivative of u in the outward normal direction of ∂K, and $d\gamma$ does the infinitesimal element of ∂K. Conventional efforts of error analysis have been focused on the estimation of the second term in the right-hand side of (3), which is absent in the conforming case. To cope with such difficulty, we introduce the lowest-order Raviart-Thomas triangular $H(\mathrm{div})$ finite element space W^h associated to each $\mathcal{T}^h$ [9, 16]. Then, noting that the normal component of $\forall q_h \in W^h$ is constant and continuous along each inter-element edge, we can derive $(q_h, \nabla_h v_h) + (\mathrm{div}\, q_h, v_h) = 0$, and hence

$$(\nabla_h u_h - \nabla u, \nabla_h v_h) = (q_h - \nabla u, \nabla_h v_h) + (\mathrm{div}\, q_h + f, v_h)\,; \;\; \forall\, q_h \in W^h,\; \forall v_h \in V^h\,. \tag{4}$$

Then by Lemma 6 of [12], a refinement of Strang's second lemma [10], we have

$$\|\nabla u - \nabla_h u_h\|^2 = \inf_{v_h\in V^h}\|\nabla u - \nabla_h v_h\|^2 + \left[\sup_{w_h\in V^h\setminus\{0\}}\frac{(q_h - \nabla u, \nabla_h w_h) + (\mathrm{div}\, q_h + f, w_h)}{\|\nabla_h w_h\|}\right]^2, \tag{5}$$

where $\|\cdot\|$ stands for the norms of both $L_2(\Omega)$ and $L_2(\Omega)^2$. Using the Fortin operator $\Pi_h^F : \; H(\mathrm{div};\Omega) \cap H^{\frac{1}{2}+\delta}(\Omega)^2 \to W^h$ $(\delta > 0)$ (cf. [9]) and the orthogonal projection one $Q_h : \; L_2(\Omega) \to X^h :=$ space of step functions over $\mathcal{T}^h$, we obtain a priori error estimate:

$$\|\nabla u - \nabla_h u_h\|^2 \le \inf_{v_h\in V^h}\|\nabla u - \nabla_h v_h\|^2 + \left[\|\nabla u - \Pi_h^F\nabla u\| + \sup_{w_h\in V^h\setminus\{0\}}\frac{(f - Q_h f, w_h - Q_h w_h)}{\|\nabla_h w_h\|}\right]^2, \tag{6}$$

where q_h in (5) is taken as $\Pi_h^F\nabla u$.
We can obtain a more concrete error estimate in terms of the mesh parameter $h_* > 0$ (h will be used in a different meaning later) by deriving estimates such as, for $\forall v \in H_0^1(\Omega)\cap H^2(\Omega)$ and $\forall g \in H^1(\Omega) + V^h$,

$$\begin{aligned} \|v - \Pi_h v\| &\le \gamma_0 h_*^2 |v|_2\,, & \|\nabla v - \nabla_h \Pi_h v\| &\le \gamma_1 h_* |v|_2\,,\\ \|\nabla v - \Pi_h^F \nabla v\| &\le \gamma_2 h_* |v|_2\,, & \|g - Q_h g\| &\le \gamma_3 h_* \|\nabla_h g\|\,, \end{aligned} \tag{7}$$

where $|\cdot|_k$ denotes the standard seminorm of $H^k(\Omega)$ $(k \in \mathbf{N})$ [10], and γ_0, γ_1, γ_2 and γ_3 are positive error constants dependent only on $\{\mathcal{T}^h\}_{h>0}$.
Then we obtain, for the solution $u \in H_0^1(\Omega)\cap H^2(\Omega)$,

$$\|\nabla u - \nabla_h u_h\| \le \begin{cases} h_*\{\gamma_1^2|u|_2^2 + (\gamma_2|u|_2 + \gamma_3\|f\|)^2\}^{1/2} & \text{for } f \in L_2(\Omega),\\ h_*\{\gamma_1^2|u|_2^2 + (\gamma_2|u|_2 + \gamma_3^2 h_*|f|_1)^2\}^{1/2} & \text{for } f \in H^1(\Omega), \end{cases} \tag{8}$$

where the term $|u|_2$ can be bounded as $|u|_2 \le \|f\|$ for the present Ω.
We can also use Nitsche's trick to evaluate a priori L_2 error of u_h [10, 17]. That is, let us define $\psi \in H_0^1(\Omega)\,(\cap H^2(\Omega))$ for $e^h := u - u_h$ by

$$(\nabla\psi, \nabla v) = (e^h, v)\,;\; \forall v \in H_0^1(\Omega)\,. \tag{9}$$

Then, for $\forall v_h \in V^h$ and $\forall q_h,\ \tilde{q}_h \in W^h$, we have

$$\|e^h\|^2 = (\tilde{q}_h - \nabla_h v_h, \nabla_h e^h) + (\nabla_h v_h - \nabla\psi, \nabla u - q_h) + (\psi - v_h, \operatorname{div} q_h + f) + (\operatorname{div} \tilde{q}_h + e^h, e^h)\,. \quad (10)$$

Substituting $v_h = \Pi_h\psi$, $q_h = \Pi_h^F \nabla u$ and $\tilde{q}_h = \Pi_h^F \nabla\psi$ above, we find

$$\begin{aligned}\|e^h\|^2 = & (\Pi_h^F \nabla\psi - \nabla\psi + \nabla\psi - \nabla_h \Pi_h \psi, \nabla_h e^h) + (\nabla_h \Pi_h \psi - \nabla\psi, \nabla u - \Pi_h^F \nabla u) \\ & + (\psi - \Pi_h\psi, f - Q_h f) + (e^h - Q_h e^h, e^h - Q_h e^h)\,,\end{aligned} \quad (11)$$

since $\operatorname{div} q_h = \operatorname{div} \Pi_h^F \nabla u = -Q_h f$ and $\operatorname{div} \tilde{q}_h = \operatorname{div} \Pi_h^F \nabla\psi = -Q_h e^h$. Then we have, by (7) as well as the relations $|u|_2 \le \|f\|$ and $|\psi|_2 \le \|e^h\|$,

$$\|e^h\|^2 \le \left[(\gamma_1 + \gamma_2) h_* \|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2) h_*^2 \|f\|\right] \|e^h\| + \gamma_3^2 h_*^2 \|\nabla_h e^h\|^2\,, \quad (12)$$

where the term $\gamma_0 h_*^2 \|f\| \cdot \|e^h\|$ can be replaced with $\gamma_0\gamma_3 h_*^3 |f|_1 \|e^h\|$ if $f \in H^1(\Omega)$. This may be considered a quadratic inequality for $\|e^h\|$, and solving it gives an expected order estimate $\|u - u_h\| = \|e^h\| = O(h_*^2)$:

$$\|e^h\| \le \frac{h_*}{2}\left(A_1 + \sqrt{A_1^2 + 4A_2}\right);\ A_1 = (\gamma_1 + \gamma_2)\|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2) h_* \|f\|,\ A_2 = \gamma_3^2 \|\nabla_h e^h\|^2. \quad (13)$$

RELATION TO RAVIART-THOMAS MIXED FEM

We have already introduced the Raviart-Thomas space W^h for auxiliary purposes. But it is well known that the present non-conforming FEM is closely related to the Raviart-Thomas mixed FEM [3, 18]. Here we will summarize the implementation of such a mixed FEM by slightly modifying the original non-conforming P_1 scheme described by (2). The original idea in [3, 18] is based on the enrichment by the conforming cubic bubble functions with the L_2 projection into W^h, but we here adopt non-conforming quadratic bubble ones to make the modification procedure a little simpler.
Firstly, we replace f in (2) by $Q_h f$. Then u_h is modified to $u_h^* \in V^h$ defined by

$$(\nabla_h u_h^*, \nabla_h v_h) = (Q_h f, v_h)\,;\ \ \forall\, v_h \in V^h\,. \quad (14)$$

Secondly, we introduce the space V_B^h of non-conforming quadratic bubble functions by defining its basis function φ_K associated to each $K \in \mathcal{T}^h$: φ_K vanishes outside K and its value at $x \in K$ is given by

$$\varphi_K(x) = \frac{1}{2}|x - x^G|^2 - \frac{1}{12}\sum_{i=1}^{3} |x^{(i)} - x^G|^2\,, \quad (15)$$

where $|\cdot|$ is the Euclidean norm of $\mathbf{R}^2$, x^G the barycenter of K, and $x^{(i)}$ for $i = 1, 2, 3$ the i-th vertex of K. It is easy to see that the line integration of φ_K for each edge e of K vanishes:

$$\int_e \varphi_K \, d\gamma = 0\,. \quad (16)$$

Now the enriched non-conforming finite element space $\tilde{V}^h$ is defined by the following linear sum:

$$\tilde{V}^h = V^h + V_B^h\,. \quad (17)$$

By (16) and the Green formula, we find the following orthogonality relation for $(\nabla_h\cdot, \nabla_h\cdot)$:

$$(\nabla_h v_h, \nabla_h \beta_h) = 0\ \ ;\ \forall\, v_h \in V^h\,,\ \ \forall\, \beta_h \in V_B^h\,. \quad (18)$$

Then the modified finite element solution $\tilde{u}_h \in \tilde{V}^h$ is defined by

$$(\nabla_h \tilde{u}_h, \nabla_h \tilde{v}_h) = (Q_h f, \tilde{v}_h)\,;\ \ \forall\, \tilde{v}_h \in \tilde{V}^h\,. \quad (19)$$

Thanks to (18), the present $\tilde{u}_h$ can be obtained as the sum

$$\tilde{u}_h = u_h^* + \alpha_h \,, \tag{20}$$

where $u_h^* \in V^h$ is the solution of (14), and $\alpha_h \in V_B^h$ is determined by

$$(\nabla_h \alpha_h, \nabla_h \beta_h) = (Q_h f, \beta_h)\,; \;\; \forall \beta_h \in V_B^h \,, \tag{21}$$

i. e., completely independently of u_h^*. Moreover, α_h can be decided by element-by-element computations. More specifically, denoting $\alpha_h|K$ as $\alpha_K \varphi_K|K$, Eq. (21) leads to

$$\alpha_K (\nabla \varphi_K, \nabla \varphi_K)_K = (Q_h f, \varphi_K)_K \,; \;\; \forall K \in \mathcal{T}^h \,, \tag{22}$$

where $(\cdot,\cdot)_K$ denotes the inner products of both $L_2(K)$ and $L_2(K)^2$.
Define $\{p_h, \overline{u}_h\} \in L_2(\Omega)^2 \times X^h$ by

$$p_h = \nabla_h \tilde{u}_h \,, \;\; \overline{u}_h = Q_h \tilde{u}_h \,. \tag{23}$$

By applying the Green formula to (19), we can show that $p_h \in W^h$, and also that the present pair $\{p_h, \overline{u}_h\}$ satisfies the determination equations of the lowest-order Raviart-Thomas mixed FEM:

$$\begin{cases} (p_h, q_h) + (\overline{u}_h, \operatorname{div} q_h) = 0 & ; \; \forall \, q_h \in W^h \,, \\ (\operatorname{div} p_h, \overline{v}_h) = -(Q_h f, \overline{v}_h) & ; \; \forall \, \overline{v}_h \in X^h \,. \end{cases} \tag{24}$$

By the uniqueness of the solutions, $\{p_h, \overline{u}_h\}$ is nothing but the unique solution of (24).
In conclusion, denoting the constant value of $Q_h f|K$ by $\overline{f}_K \;\left(= \int_K f\, dx / \operatorname{meas}(K)\right)$, we have for $\forall \, K \in \mathcal{T}^h$ and $\forall x \, \in K$ that

$$\alpha_K = -\frac{1}{2}\overline{f}_K \,, \;\; \tilde{u}_h(x) = u_h^*(x) + \alpha_K \varphi_K(x) = u_h^*(x) - \frac{1}{4}\overline{f}_K \big(|x - x^G|^2 - \frac{1}{6}\sum_{i=1}^{3} |x^{(i)} - x^G|^2\big) \,,$$

$$p_h(x) = \nabla u_h^*(x) - \frac{1}{2}\overline{f}_K (x - x^G) \,, \quad \overline{u}_h(x) = u_h^*(x^G) - \frac{1}{16}\overline{f}_K \big(|x^G|^2 - \frac{1}{3}\sum_{i=1}^{3} |x^{(i)}|^2\big) \,, \tag{25}$$

which coincide with those in [18] and are easy to compute by post-processing.

A POSTERIORI ERROR ESTIMATION

The consideration in the preceding section suggests the a posteriori error estimation based on the hypercircle method [11, 16].
Taking notice of the fact that $p_h \in W^h$ obtained in the preceding section belongs to $H(\operatorname{div};\Omega)$ with $\operatorname{div} p_h = -Q^h f$, we find that, for $\forall v \in H_0^1(\Omega)$,

$$\|\nabla v - p_h\|^2 = \|\nabla (v - u^h)\|^2 + \|\nabla u^h - p_h\|^2 \,, \quad \|\nabla u^h - \frac{1}{2}(\nabla v + p_h)\| = \frac{1}{2}\|\nabla v - p_h\| \,, \tag{26}$$

where $u^h \in H_0^1(\Omega)$ is the solution of (1) with f replaced by $Q_h f$:

$$(\nabla u^h, \nabla v) = (Q_h f, v)\,; \;\; \forall \, v \in H_0^1(\Omega) \,. \tag{27}$$

Eq. (26) implies that the three points ∇u^h, ∇v and p_h in $L_2(\Omega)^2$ make a hypercircle, the first having a right inscribed angle. Noting that $(f - Q_h f, v) = (f - Q_h f, v - Q_h v)$ for $\forall v \in H_0^1(\Omega) \subset L_2(\Omega)$, we have by (7) that

$$|u - u^h|_1 = \|\nabla (u - u^h)\| \le \gamma_3 h_* \|f - Q_h f\| \quad \left(\le \gamma_3^2 h_*^2 |f|_1 \text{ if } f \in H^1(\Omega)\right) \,. \tag{28}$$

Taking an appropriate $v \in H_0^1(\Omega)$, we obtain a posteriori error estimates related to $p_h = \nabla_h \tilde{u}_h$:

$$\|\nabla u - p_h\| \le \|\nabla v - p_h\| + \|\nabla(u-u^h)\|\,, \quad \|\nabla u - \frac{1}{2}(\nabla v + p_h)\| \le \frac{1}{2}\|\nabla v - p_h\| + \|\nabla(u-u^h)\|\,. \quad (29)$$

A typical example of v is the conforming P_1 finite element solution $u_h^C \in V_C^h$, where V_C^h is the conforming P_1 space over $\mathcal{T}^h$. Another example is a function $v_h^C \in V_C^h$ obtained by appropriate post-processing of u_h or u_h^*, such as nodal averaging or smoothing. A cheap method of constructing a nice v_h^C may be an interesting subject. Again we need the constant γ_3 to evaluate the term $\|\nabla(u-u^h)\|$ above.
If we use $\nabla_h u_h$ based on the original $u_h \in V^h$ in (2), instead of $\tilde{u}_h \in \tilde{V}_h$, we must evaluate some additional terms. Fortunately, such evaluation can be done explicitly by γ_3 and some constants related to $\{\varphi_K\}_{K\in\mathcal{T}^h}$. The error $\|\nabla u - \nabla_h u_h\|$ thus evaluated can be also used to give a posteriori L_2 estimate based on (13).

ERROR CONSTANTS

To analyze the error constants in (7), let us consider their elementwise counterparts. Let h, α and θ be positive constants such that

$$h > 0\,, \quad 0 < \alpha \le 1\,, \quad (\frac{\pi}{3} \le) \cos^{-1}\frac{\alpha}{2} \le \theta < \pi\,. \quad (30)$$

Then we define the triangle $T_{\alpha,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$ and $B(\alpha h\cos\theta, \alpha h\sin\theta)$. From (30), AB is shown to be the edge of maximum length, i. e. $\overline{AB} \ge h \ge \alpha h$, so that $h = \overline{OA}$ here denotes the medium edge length, unlike the usual usage as the largest one [10]. A point on the closure $\overline{T}_{\alpha,\theta,h}$ of $T_{\alpha,\theta,h}$ is denoted by $x = \{x_1, x_2\}$, and the three edges e_i's (i=1,2,3) are defined by $\{e_1, e_2, e_3\} = \{OA, OB, AB\}$. By an appropriate congruent transformation in $\mathbf{R}^2$, we can configure any triangle as $T_{\alpha,\theta,h}$. As the usage in [5], we will use abbreviated notations $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$ and $T = T_1$ (Fig. 1). We will also use the notations $\|\cdot\|_{T_{\alpha,\theta,h}}$ and $|\cdot|_{k,T_{\alpha,\theta,h}}$ as the norm of $L_2(T_{\alpha,\theta,h})$ and seminorms of $H^k(T_{\alpha,\theta,h})$ $(k\in\mathbf{N})$, where the subscript $T_{\alpha,\theta,h}$ will be usually omitted.

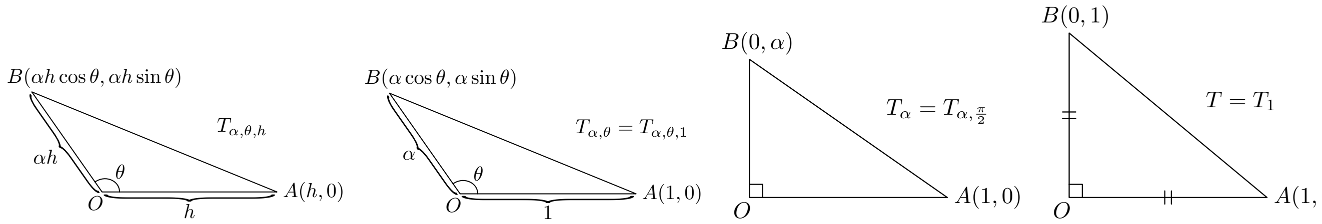

Figure 1: Notations for triangles : $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$, $T = T_1$

Let us define the following closed linear spaces for functions over $T_{\alpha,\theta,h}$:

$$V^0_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) \,|\, \int_{T_{\alpha,\theta,h}} v(x)\,dx = 0\}, \quad (31)$$

$$V^i_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) | \int_{e_i} v(s)\,ds = 0\} \ (i=1,2,3), \quad (32)$$

$$V^{\{1,2\}}_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) | \int_{e_1} v(s)\,ds = \int_{e_2} v(s)\,ds = 0\}, \quad (33)$$

$$V^{\{1,2,3\}}_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) | \int_{e_i} v(s)\,ds = 0 \ (i=1,2,3)\}, \quad (34)$$

$$V^4_{\alpha,\theta,h} = \{v \in H^2(T_{\alpha,\theta,h}) | \int_{e_i} v(s)\,ds = 0 \ (i=1,2,3)\}. \quad (35)$$

For the above, we will again use abbreviated notations like $V^0_{\alpha,\theta} = V^0_{\alpha,\theta,1}$, $V^0_\alpha = V^0_{\alpha,\pi/2}$, $V^0 = V^0_1$ etc.
Let us consider the (elementwise) P_0 interpolation operator $\Pi^0_{\alpha,\theta,h}$ and non-conforming P_1 one $\Pi^{1,N}_{\alpha,\theta,h}$ for functions on $T_{\alpha,\theta,h}$ [8, 10] : $\Pi^0_{\alpha,\theta,h}v$ for $\forall v \in H^1(T_{\alpha,\theta,h})$ is a constant function such that

$$(\Pi^0_{\alpha,\theta,h}v)(x) = \int_{T_{\alpha,\theta,h}} v(y)\,dy \Big/ \int_{T_{\alpha,\theta,h}} dy \quad (\forall x \in T_{\alpha,\theta,h}), \quad (36)$$

while $\Pi^{1,N}_{\alpha,\theta,h}v$ for $\forall v \in H^1(T_{\alpha,\theta,h})$ is a linear function such that

$$\int_{e_i} (\Pi^{1,N}_{\alpha,\theta,h}v)(s)\,ds = \int_{e_i} v(s)\,ds \quad \text{for} \quad i = 1, 2, 3\,. \tag{37}$$

To analyze these interpolation operators, let us estimate the positive constants defined by

$$C_J(\alpha,\theta,h) = \sup_{v \in V^J_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_1} \quad (J = 0,1,2,3,\{1,2\},\{1,2,3\})\,, \tag{38}$$

$$C_4(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{|v|_1}{|v|_2}\,,\quad C_5(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_2}\,. \tag{39}$$

We will again use abbreviated notations $C_J(\alpha,\theta) = C_J(\alpha,\theta,1)$, $C_J(\alpha) = C_J(\alpha,\pi/2)$, and $C_J = C_J(1)$. By a simple scale change, we find that $C_J(\alpha,\theta,h) = hC_J(\alpha,\theta)$ $(J \neq 5)$ and $C_5(\alpha,\theta,h) = h^2C_5(\alpha,\theta)$. Now, by noting $v - \Pi^0_{\alpha,\theta,h}v \in V^0_{\alpha,\theta,h}$ for $v \in H^1(T_{\alpha,\theta,h})$ and $v - \Pi^{1,N}_{\alpha,\theta,h}v \in V^4_{\alpha,\theta,h}$ for $v \in H^2(T_{\alpha,\theta,h})$, we can easily have the popular type of interpolation error estimates on $T_{\alpha,\theta,h}$ [8, 10]:

$$\|v - \Pi^0_{\alpha,\theta,h}v\| \le C_0(\alpha,\theta)h|v|_1\,;\ \forall v \in H^1(T_{\alpha,\theta,h}), \tag{40}$$

$$|v - \Pi^{1,N}_{\alpha,\theta,h}v|_1 \le C_4(\alpha,\theta)h|v|_2\,;\ \ \forall v \in H^2(T_{\alpha,\theta,h}), \tag{41}$$

$$\|v - \Pi^{1,N}_{\alpha,\theta,h}v\| \le C_5(\alpha,\theta)h^2|v|_2\,;\ \forall v \in H^2(T_{\alpha,\theta,h}). \tag{42}$$

We can show that the following relations hold for the above constants:

$$C_4(\alpha,\theta) \le C_0(\alpha,\theta)\,,\ \ C_5(\alpha,\theta) \le C_0(\alpha,\theta)C_{\{1,2,3\}}(\alpha,\theta) \le C_0(\alpha,\theta)C_{\{1,2\}}(\alpha,\theta)\,. \tag{43}$$

An estimation rougher than the latter of (43) is $C_5(\alpha,\theta) \le C_0(\alpha,\theta)\min_{i=1,2,3} C_i(\alpha,\theta)$. To show the former of (43), we first derive $\int_{T_{\alpha,\theta}} \partial v/\partial x_i\,dx = 0$ for $\forall v \in V^4_{\alpha,\theta}$ $(i = 1,\,2)$ by using (35) and the Gauss formula. Then we can easily obtain the desired result by noting the definition of $C_0(\alpha,\theta)$. To derive the latter of (43), we should evaluate $\|v\|/|v|_1$ and $|v|_1/|v|_2$ for $\forall v \in V^4_{\alpha,\theta}$ $(i = 1,\,2)$. The former quotient can be evaluated by using $C_{\{1,2,3\}}(\alpha,\theta)$, while the latter can be done by $C_4(\alpha,\theta)$ and the former of (43). Clearly, $C_{\{1,2,3\}}(\alpha,\theta) \le C_{\{1,2\}}(\alpha,\theta)$, and we have the latter of (43).

Thus we can give quantitative interpolation estimates from (40) through (42), if we succeed in evaluating or bounding the constants $C_J(\alpha,\theta)$'s explicitly for all possible J. Among them, $C_0(\alpha,\theta)$ and $C_{\{1,2\}}(\alpha,\theta)$ are important as may be seen from (43). Notice that each of such constants can be characterized by minimization of a kind of Rayleigh quotient [5, 20, 21]. Then it is equivalent to finding the minimum eigenvalue of a certain eigenvalue problem expressed by a weak formulation for a partial differential equation with some auxiliary conditions.

Moreover, we already derived some results for $C_i(\alpha,\theta)$ for $i = 0,\,1,\,2$ [14, 15]. In particular, $C_0 = 1/\pi$, and $C_1(= C_2)$ is equal to the maximum positive solution of the equation $1/\mu + \tan(1/\mu) = 0$ for μ. The constants $C_J(\alpha,\theta)$'s for $J = 0,1,2,3,4,5,\{1,2\},\{1,2,3\}$ are bounded uniformly for $\{\alpha,\theta\}$. More specifically, their explicit upper bounds are given in terms of α, θ and their values at $\{\alpha,\theta\} = \{1,\pi/2\}$. Furthermore, $C_J(\alpha)$'s except for $J = 4$ are monotonically increasing in α. Asymptotic behaviors of the constants $C_J(\alpha)$'s for $\alpha \downarrow 0$ can be also analyzed [15]. As a result, the interpolation by the non-conforming P_1 triangle is robust to the distortion of $T_{\alpha,\theta}$. This fact does not necessarily imply the robustness of the final error estimates for $u - u_h$, since analysis of the Fortin interpolation has not been performed yet.

Remark 1. Instead of $\Pi^{1,N}_{\alpha,\theta,h}$, it is also possible to consider an interpolation operator using the function values at midpoints of edges. Such an operator is definable for continuous functions over $\overline{T}_{\alpha,\theta,h}$, but not so for functions in $H^1(T_{\alpha,\theta,h})$. Moreover, its analysis would be different from that for $\Pi^{1,N}_{\alpha,\theta,h}$.

DETERMINATION OF $C_{\{1,2\}}$

From the preceding observations, we can give explicit upper bounds of various interpolation constants associated to the non-conforming P_1 triangle, provided that the value of $C_{\{1,2\}}$ is determined. This becomes indeed possible by adopting essentially the same idea and techniques to determine C_0 and C_1 $(= C_2)$:

Theorem 1. $C_{\{1,2\}} = C_{\{1,2\}}(1, \pi/2, 1)$ *is equal to the maximum positive solution of the transcendental equation for* μ*:*

$$\frac{1}{2\mu} + \tan\frac{1}{2\mu} = 0\,. \tag{44}$$

The above implies that $C_{\{1,2\}} = \frac{1}{2}C_1 (= \frac{1}{2}C_2)$*, and hence is bounded as, with numerical verification,*

$$0.24641 < C_{\{1,2\}} < 0.24647\,. \tag{45}$$

Remark 2. Thus 1/4 is a simple but nice upper bound. Numerically, we have $C_{\{1,2\}} = 0.2464562258\cdots$.

Proof. By the use of the techniques for determination of C_0 and $C_1 = C_2$ in [14, 15], we obtain the following equation for μ:

$$1 + \frac{1}{2\mu}\sin\frac{1}{\mu} - \cos\frac{1}{\mu} = 0,$$

whose maximum positive solution is the desired $C_{\{1,2\}}$. By the double-angle formulas, the above is transformed into

$$(2\sin\frac{1}{2\mu} + \frac{1}{\mu}\cos\frac{1}{2\mu})\sin\frac{1}{2\mu} = 0.$$

It is now easy to derive (44), and also to draw other conclusions by using the results in [14, 15]. □

ANALYSIS OF FORTIN'S INTERPOLATION

This section is devoted to analysis of the Fortin interpolation operator $\Pi^F_{\alpha,\theta}$ for each $T_{\alpha,\theta}$ [9]. First, let us introduce the following transformation between $x = \{x_1, x_2\} \in T_{\alpha,\theta}$ and $\hat{x} = \{\hat{x}_1, \hat{x}_2\}$:

$$\hat{x}_1 = x_1\sin\theta - x_2\cos\theta\,,\ \ \hat{x}_2 = x_1\cos\theta + x_2\sin\theta\,. \tag{46}$$

For each $q = \{q_1, q_2\} \in H(\mathrm{div}; T_{\alpha,\theta})$, we also consider the (contravariant) expression $\hat{q} = \{\hat{q}_1, \hat{q}_2\}$:

$$\hat{q}_1 = q_1\sin\theta - q_2\cos\theta\,,\ \ \hat{q}_2 = q_1\cos\theta + q_2\sin\theta\,, \tag{47}$$

for which we loosely use both x and $\hat{x}$ as variables. The Raviart-Thomas type approximate function $q_h = \{q_{h1}, q_{h2}\}$ are given, together with the expression for $\hat{q}_h = \{\hat{q}_{h1}, \hat{q}_{h2}\}$, by

$$\left\{\begin{array}{l} q_{h1} = \alpha_1 + \alpha_3 x_1 \\ q_{h2} = \alpha_2 + \alpha_3 x_2 \end{array}\right. ,\quad \left\{\begin{array}{l} \hat{q}_{h1} = \alpha_1\sin\theta - \alpha_2\cos\theta + \alpha_3\hat{x}_1 \\ \hat{q}_{h2} = \alpha_1\cos\theta + \alpha_2\sin\theta + \alpha_3\hat{x}_2 \end{array}\right. . \tag{48}$$

The Fortin interpolation $q^*_h = \{q^*_{h1}, q^*_{h2}\} = \Pi^F_{\alpha,\theta} q$ for $q \in H(\mathrm{div}; T_{\alpha,\theta}) \cap H^{\frac{1}{2}+\delta}(T_{\alpha,\theta})^2$ $(\delta > 0)$ is of the form of q_h in (48) and characterized by the conditions:

$$\int_{e_1}(q^*_{h2} - q_2)\,ds = \int_{e_2}(\hat{q}^*_{h1} - \hat{q}_1)\,ds = 0\,,\ \ \int_{T_{\alpha,\theta}}\mathrm{div}(q^*_h - q)\,dx = 0\,, \tag{49}$$

where $\hat{q}$ for q and $\hat{q}^*_h$ for q^*_h are defined by the relations in (47) and (48), respectively.
Let us now introduce another interpolation $\Pi^{\{1,2\}}_{\alpha,\theta} q = q^\dagger_h = \{q^\dagger_{h1}, q^\dagger_{h2}\}$ for the same q, which is a constant vector function that satisfies only the former two conditions of (49). Then we have the L_2 estimate

$$\|q - \Pi^F_{\alpha,\theta}q\| \le \|q - \Pi^{\{1,2\}}_{\alpha,\theta}q\| + \frac{\|\mathrm{div}\, q\|}{2\sqrt{|T_{\alpha,\theta}|}}\sqrt{\int_{T_{\alpha,\theta}}|x|^2 dx} = \|\Pi^{\{1,2\}}_{\alpha,\theta}q - q\| + \sqrt{\frac{1+\alpha\cos\theta+\alpha^2}{24}}\|\mathrm{div}\, q\|. \tag{50}$$

To bound $\|q - \Pi^{\{1,2\}}_{\alpha,\theta}q\|$, let us evaluate $\|\hat{q}_1 - \hat{q}^\dagger_{h1}\|$ and $\|q_2 - q^\dagger_{h2}\|$ by using $C_1(\alpha,\theta)$ and $C_2(\alpha,\theta)$:

Theorem 2. *It holds for* $q = \{q_1, q_2\} \in H^1(T_{\alpha,\theta})^2$ *that*

$$\|q - \Pi^{\{1,2\}}_{\alpha,\theta} q\| \leq C_6(\alpha,\theta)|q|_1\,; \;\; C_6(\alpha,\theta) := \frac{1}{\sqrt{2}\sin\theta}\Big\{C^2_{1,\alpha,\theta} + C^2_{2,\alpha,\theta} + 2C_{1,\alpha,\theta}C_{2,\alpha,\theta}\cos^2\theta$$
$$+ (C_{1,\alpha,\theta} + C_{2,\alpha,\theta})\sqrt{C^2_{1,\alpha,\theta} + C^2_{2,\alpha,\theta} + 2C_{1,\alpha,\theta}C_{2,\alpha,\theta}\cos 2\theta}\Big\}^{1/2}, \quad (51)$$

where $C_{i,\alpha,\theta} = C_i(\alpha,\theta)\ (i = 1, 2)$, *and* $|q|_1 = \sqrt{|q_1|_1^2 + |q_2|_1^2}$.

Remark 3. From (50) and (51), it is easy to derive the following estimate for the Fortin interpolation operator $\Pi^F_{\alpha,\theta,h}$ for $T_{\alpha,\theta,h}$:

$$\|q-\Pi^F_{\alpha,\theta,h}q\| \leq C_6(\alpha,\theta)h|q|_1 + C_7(\alpha,\theta)h\|\mathrm{div}\, q\|\,; \; C_7(\alpha,\theta) := \sqrt{\frac{1+\alpha\cos\theta+\alpha^2}{24}},\ \forall q \in H^1(T_{\alpha,\theta,h})^2. \quad (52)$$

Because of the factor $\sin\theta$ in (51), the maximum angle condition applies to estimate (51), and hence to (52) [1, 5, 17]. On the other hand, the estimates for $\Pi^0_{\alpha,\theta,h}$ and $\Pi^{1,N}_{\alpha,\theta,h}$ are free from such conditions as may be seen from (43) and the comments there.

GLOBAL INTERPOLATION OPERATORS

So far, we have introduced and analyzed local interpolation operators $\Pi^0_{\alpha,\theta,h}$, $\Pi^{1,N}_{\alpha,\theta,h}$ and $\Pi^F_{\alpha,\theta,h}$. For each $K \in \mathcal{T}^h$, we can find an appropriate $T_{\alpha,\theta,h}$ congruent to K under a mapping $\Phi_K : K \to T_{\alpha,\theta,h}$. Then it is natural to define the P_1 non-conforming interpolation operator $\Pi_h : H^1_0(\Omega) \to V^h$ by $\Pi_h u|K = [\Pi^{1,N}_{\alpha,\theta,h}(v|K \circ \Phi_K^{-1})] \circ \Phi_K$ for $\forall v \in H^1_0(\Omega)$ and $\forall K \in \mathcal{T}^h$. Similarly, the orthogonal projection operator $Q_h : L_2(\Omega) \to X^h$ is related to $\Pi^0_{\alpha,\theta,h}$, while the global Fortin operator Π^F_h is defined through $\Pi^F_{\alpha,\theta,h}$, Φ_K and the Piola transformation for 2D contravariant vector fields [9].
For each $K \in \mathcal{T}^h$, define $\{\alpha_K, \theta_K, h_K\}$ as $\{\alpha, \theta, h\}$ of the associated $T_{\alpha,\theta,h}$. Then, our analysis shows that the estimates in (7) can be concretely given by, for $\forall v \in H^1_0(\Omega) \cap H^2(\Omega)$ and $\forall g \in H^1(\Omega) + V^h$,

$$\|v - \Pi_h v\| \leq C^h_5 h^2_*|v|_2 \leq C^h_0 C^h_{\{1,2\}} h^2_*|v|_2\,, \qquad \|\nabla v - \nabla_h \Pi_h v\| \leq C^h_4 h_*|v|_2 \leq C^h_0 h_*|v|_2\,,$$
$$\|\nabla v - \Pi^F_h \nabla v\| \leq C^h_6 h_*|v|_2 + C^h_7 h_*\|\Delta v\|\,, \qquad \|g - Q_h g\| \leq C^h_0 h_*\|\nabla_h g\|\,, \quad (53)$$

where

$$h_* = \max_{K\in\mathcal{T}^h} h_K\,, \quad C^h_J = \max_{K\in\mathcal{T}^h} C_J(\alpha_K, \theta_K) \quad (J = 0, 4, 5, 6, 7, \{1, 2\})\,. \quad (54)$$

Remark 4. Relations such as (20), (23) and (25) may suggest the possibility of finding interpolations for ∇u in W^h better than that by the Fortin operator, which are free from the maximum angle condition [5]. However, $\nabla_h(\Pi_h u + \alpha_h)$, for example, is not shown to belong to W^h, because we cannot prove the inter-element continuity of normal components unlike $\nabla_h \tilde{u}_h$. Our numerical results show that such a condition is probably essential for the non-conforming P_1 triangle. See also [1] for related topics.

NUMERICAL RESULTS

Firstly, we performed numerical computations to see the actual dependence of various constants on α and θ by adopting the conforming P_1 element and a kind of discrete Kirchhoff plate bending element [13], the latter of which is used to deal with directly the 4-th order partial differential eigenvalue problems related to $C_4(\alpha,\theta)$ and $C_5(\alpha,\theta)$. That is, we obtained some numerical results for $C_4(\alpha)$ and $C_5(\alpha)$ ($\theta = \pi/2$) together with their upper bounds. We used the uniform triangulations of the entire domain T_α: T_α is subdivided into small triangles, all being congruent to $T_{\alpha,\pi/2,h}$ with e. g. $h = 1/20$.
The left-hand side of Fig. 2 illustrates the graphs of approximate $C_4(\alpha)$ and $C_0(\alpha)$ versus $\alpha \in]0, 1]$, while the right-hand side does similar graphs for $C_5(\alpha)$ and $C_0(\alpha)C_{\{1,2\}}(\alpha)$. In both cases, the theoretical upper bounds based on (43) give fairly good approximations to the considered constants $C_4(\alpha)$ and $C_5(\alpha)$. Asymptotic behaviors of the constants for $\alpha \downarrow 0$ observed in the figures can be analyzed as in [15].

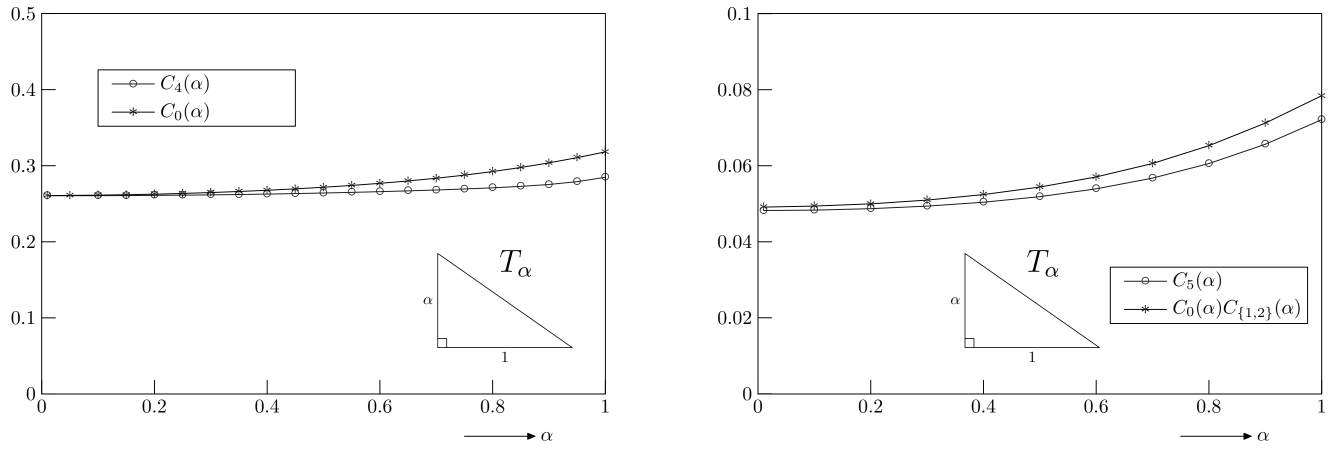

Figure 2: Numerical results for $C_4(\alpha)$ & $C_0(\alpha)$ (left), and for $C_5(\alpha)$ & $C_0(\alpha)C_{\{1,2\}}(\alpha)$ (right) ; $0 < \alpha \le 1$

We also tested numerically the validity of our a priori error estimate for $\|\nabla u - \nabla_h u_h\|$. That is, we choose Ω as the unit square $\{x = \{x_1, x_2\};\ 0 < x_1,\ x_2 < 1\}$and f as $f(x_1, x_2) = \sin \pi x_1 \sin \pi x_2$, and consider the $N \times N$ Friedrichs-Keller type uniform triangulations $(N \in \mathbf{N})$. In such situation, $u(x_1, x_2) = \dfrac{1}{2\pi^2} \sin \pi x_1 \sin \pi x_2$, and all the triangles are congruent to a right isosceles triangle $T_{1,\pi/2,1/N}$, i. e., $h_* = h = 1/N$. Moreover, we can use the following values or upper bounds for necessary constants:

$$C_0^h = C_0 = 1/\pi\,, \quad C_{\{1,2\}}^h = C_{\{1,2\}} \lessapprox 1/4\,, \quad C_6^h = C_1 = C_2 \lessapprox 1/2\,, \quad C_7^h = C_7 = 1/\sqrt{12}\,. \tag{55}$$

Figure 3 illustrates the comparison of the actual $\|\nabla u - \nabla_h u_h\|$ and its a priori estimate based on our analysis. The difference is still large, but anyway our analysis appears to give correct upper bounds and order of errors. Probably, a posteriori estimation mentioned previously would give more realistic results.

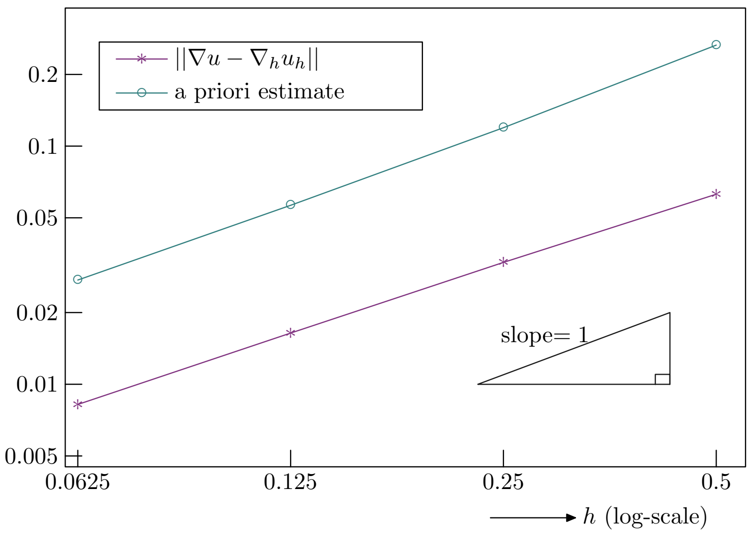

Figure 3: Numerical results for $\|\nabla u - \nabla_h u_h\|$ and its a priori estimate versus h

CONCLUDING REMARKS

We have obtained some theoretical and numerical results for several error constants associated to the non-conforming P_1 triangle, which we hope to be effectively used in quantitative error estimates, which are necessary for adaptive mesh refinements [7] and numerical verifications. Especially for numerical verification of partial differential equations by Nakao's method [19], accurate bounding of various error constants is essential. Moreover, we are planning to extend our analysis to its 3D counterpart, i. e., the non-conforming P_1 tetrahedron with face DOF's.

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