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Estimation of interpolation error constants for the P_0 and P_1 triangular finite elements

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Dedicated to Professor Ivo Babuška on the occasion of his 80th birthday.

Abstract

We give some fundamental results on the error constants for the piecewise constant interpolation function and the piecewise linear one over triangles. For the piecewise linear one, we mainly analyze the conforming case, but some results are also given for the non-conforming case. We obtain explicit relations for the dependence of such error constants on the geometric parameters of triangles. In particular, we explicitly determine the Babuška–Aziz constant, which plays an essential role in the interpolation error estimation of the linear triangular finite element. The equation for determination is the transcendental equation $\sqrt{\lambda} + \tan\sqrt{\lambda} = 0$, so that the solution can be numerically obtained with desired accuracy and verification. Such highly accurate approximate values for the constant as well as estimates for other constants can be widely used for a priori and a posteriori error estimations in adaptive computation and numerical verification of finite element solutions.

MSC: 65N15; 65N30

Keywords: FEM; Error estimates; Triangular element; Babuška–Aziz constant; Interpolation error constants

1. Introduction

The finite element method (FEM) is now recognized as a powerful numerical method for wide classes of partial differential equations. Furthermore, it also has sound mathematical bases such as highly refined a priori and a posteriori error estimations. In the classical a priori error analysis of FEM, interpolation errors are essential to derive final error estimates in various norms [7,8,10]. In this process, there appear various positive constants besides the standard discretization parameter h and norms (or semi-norms), but it has been very difficult to evaluate such constants explicitly. For quantitative purposes, however, it is indispensable to evaluate or bound them as accurately as possible, because sharper estimates enable more efficient finite element computations. Thus such an evaluation has become progressively more important and has been attempted especially for adaptive finite element calculations based on a posteriori error estimation as well as for numerical verification by FEM [1,4,6,7,13]. In this paper, we will give a few fundamental results on some interpolation error constants of the most popular triangular finite elements.

More specifically, we give some results on interpolation error constants appearing in the popular P_0 (piecewise constant) and P_1 (piecewise linear) triangular finite elements. Essentially based on the paper of Babuška–Aziz [3], we analyze the dependence of several constants on the geometric parameters such as the maximum interior angle and the minimum edge length of the triangle more quantitatively than in [3]. Above all, the optimal constant (C_3 in this

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paper) appearing in the H^1 error estimate of the P_1 interpolation of H^2 functions over the unit isosceles right triangle is essential and frequently used, and it was explicitly evaluated firstly by Natterer [15]. On the other hand, this constant was shown to be closely related to the one (C_1 in this paper) presented and effectively used by Babuška and Aziz in conjunction with the maximum angle condition [3]. More precisely, C_1 gives an upper bound quite close to the optimal constant C_3, and the relation between C_3 and C_1 was further discussed in [13,18]. Thus a precise estimation of these two constants is very important, and a number of researchers have given bounds for these using various approximation methods including numerical verification, see e.g. [2,11,13–15,18]. Furthermore, these constants can be also used to evaluate the interpolation error constants for the non-conforming P_1 triangle, as will be mentioned later.

For the above Babuška–Aziz constant, we have succeeded in obtaining a value which is in a sense optimal. That is, by analytically solving an eigenvalue problem for the 2D Laplacian over the above triangular domain, we can show that the constant can be easily determined from a solution of the simple transcendental equation $\sqrt{\lambda} + \tan\sqrt{\lambda} = 0$. In this process, we use the reflection (or symmetry) method [16]. Moreover, we have obtained some explicit relations for the dependence of such constants on the geometry of triangles. It is to be emphasized that they are consistent with the maximum angle condition in [3]. We also give some numerical and analytical results, the latter of which are based on asymptotic analysis. Thus our results can be effectively used in the quantitative a priori and a posteriori error estimations of the finite element solutions by the P_1 triangular element and also those based on the P_0 triangle. The former is of course the most classical and fundamental one, but still in frequent use, while the latter appears in some mixed finite element methods and implicitly on various occasions. Moreover, we also give some results for the non-conforming P_1 triangle by using the constants for the P_0 and the conforming P_1 triangles.

2. Preliminaries

Let h, α and θ be positive constants such that

$$h > 0, \quad 0 < \alpha \leqslant 1, \quad \left(\frac{\pi}{3} \leqslant\right) \cos^{-1}\frac{\alpha}{2} \leqslant \theta < \pi. \tag{1}$$

Then we define the triangle $T_{\alpha,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$ and $B(\alpha h\cos\theta, \alpha h \sin\theta)$. From (1), AB is shown to be the edge of maximum length, i.e. $\overline{AB} \geqslant h \geqslant \alpha h$, so that $h = \overline{OA}$ here denotes the medium edge length, although the notation h is often used as the largest edge length. A point on the closure of $T_{\alpha,\theta,h}$ is denoted by $x = \{x_1, x_2\}$. By an appropriate congruent transformation in $\mathbf{R}^2$, we can configure any triangle as $T_{\alpha,\theta,h}$. As the usage in [3], we will use abbreviated notations $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$ and $T = T_1$ (Fig. 1). Let us denote

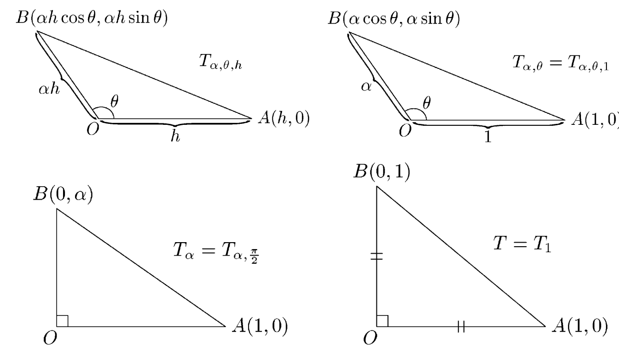


Fig. 1. Notations for triangles: $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$, $T = T_1$.

the norm of $L_2(T_{\alpha,\theta,h})$ by $\|\cdot\|_{T_{\alpha,\theta,h}}$, where the subscript $T_{\alpha,\theta,h}$ is often omitted.

Let us define the following closed linear spaces for functions over $T_{\alpha,\theta,h}$:

$$V^0_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \,\middle|\, \int_{T_{\alpha,\theta,h}} v(x)\,\mathrm{d}x = 0 \right\}, \tag{2}$$

$$V^1_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \,\middle|\, \int_0^h v(x_1, 0)\,\mathrm{d}x_1 = 0 \right\}, \tag{3}$$

$$V^2_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \,\middle|\, \int_0^h v(s\alpha\cos\theta, s\alpha\sin\theta)\,\mathrm{d}s = 0 \right\}, \tag{4}$$

$$V^3_{\alpha,\theta,h} = \{ v \in H^2(T_{\alpha,\theta,h}) | v(O) = v(A) = v(B) = 0 \}, \tag{5}$$

where $H^1(T_{\alpha,\theta,h})$ and $H^2(T_{\alpha,\theta,h})$ are respectively the first- and second-order Sobolev spaces for real square integrable functions over $T_{\alpha,\theta,h}$. For the above four spaces, we will again use abbreviated notations $V^i_{\alpha,\theta} = V^i_{\alpha,\theta,1}$, $V^i_\alpha = V^i_{\alpha,\pi/2}$ and $V^i = V^i_1$ $(0 \leqslant i \leqslant 3)$. Moreover, the spaces of constant functions and (at most) linear functions over $T_{\alpha,\theta,h}$ are respectively denoted by $\mathscr{P}_0$ and $\mathscr{P}_1$.

Let us consider the usual P_0 interpolation operator $\Pi^0_{\alpha,\theta,h}$ and P_1 one $\Pi^1_{\alpha,\theta,h}$ for functions on $T_{\alpha,\theta,h}$ [7,8,10]: $\Pi^0_{\alpha,\theta,h}v$ for $\forall v \in H^1(T_{\alpha,\theta,h})$ is a function in $\mathscr{P}_0$ well defined by

$$(\Pi^0_{\alpha,\theta,h}v)(x) = \int_{T_{\alpha,\theta,h}} v(y)\,\mathrm{d}y \Big/ \int_{T_{\alpha,\theta,h}} \mathrm{d}y \quad (\forall x \in T_{\alpha,\theta,h}), \tag{6}$$

while $\Pi^1_{\alpha,\theta,h}v$ for $\forall v \in H^2(T_{\alpha,\theta,h})$ is a function in $\mathscr{P}_1$ such that

$$(\Pi^1_{\alpha,\theta,h}v)(x) = v(x) \quad \text{for } x = O, A, B. \tag{7}$$

To give error estimates for these interpolation operators, it is natural to evaluate the positive constants defined by

$$C_i(\alpha,\theta,h) = \sup_{v \in V^i_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{\|Dv\|} \quad (i = 0, 1, 2), \tag{8}$$

$$C_3(\alpha,\theta,h) = \sup_{v \in V^3_{\alpha,\theta,h}\setminus\{0\}} \frac{\|Dv\|}{\|D^2v\|}, \quad C_4(\alpha,\theta,h) = \sup_{v \in V^3_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{\|D^2v\|}, \tag{9}$$

where $\|Dv\| = (\sum_{i=1}^2 \|\partial v/\partial x_i\|^2)^{1/2}$, and $\|D^2v\| = (\sum_{i,j=1}^2 \|\partial^2 v/\partial x_i \partial x_j\|^2)^{1/2}$. The existence of these constants easily follows from the standard compactness arguments. We will again use abbreviated notations $C_i(\alpha,\theta)=C_i(\alpha,\ \theta,1)$, $C_i(\alpha)=C_i(\alpha,\pi/2)$ and $C_i=C_i(1)$ for $0\leqslant i\leqslant 4$.

By a simple scale change, we find that $C_i(\alpha,\theta,h)=hC_i(\alpha,\theta)$ $(i=0,1,2,3)$ and $C_4(\alpha,\theta,h)=h^2C_4(\alpha,\theta)$. These relations and constants are used to evaluate interpolation errors for functions on $T_{\alpha,\theta,h}$. That is, we can easily have the popular interpolation error estimates [7,8,10]:

$$\|v-\Pi^0_{\alpha,\theta,h}v\|\leqslant C_0(\alpha,\theta)h\|Dv\|;\quad \forall v\in H^1(T_{\alpha,\theta,h}),\tag{10}$$

$$\|D(v-\Pi^1_{\alpha,\theta,h}v)\|\leqslant C_3(\alpha,\theta)h\|D^2v\|;\quad \forall v\in H^2(T_{\alpha,\theta,h}),\tag{11}$$

$$\|v-\Pi^1_{\alpha,\theta,h}v\|\leqslant C_4(\alpha,\theta)h^2\|D^2v\|;\quad \forall v\in H^2(T_{\alpha,\theta,h}),\tag{12}$$

where we have used the fact that $v-\Pi^0_{\alpha,\theta,h}v\in V^0_{\alpha,\theta,h}$ for $v\in H^1(T_{\alpha,\theta,h})$ and $v-\Pi^1_{\alpha,\theta,h}v\in V^3_{\alpha,\theta,h}$ for $v\in H^2(T_{\alpha,\theta,h})$. Moreover, for the partial derivatives of $v\in T_{\alpha,\pi/2,h}$ ($\theta=\pi/2$; right triangle case), we have

$$\left\|\frac{\partial(v-\Pi^1_{\alpha,\pi/2,h}v)}{\partial x_i}\right\|\leqslant C_i(\alpha)h\left\|D\left(\frac{\partial v}{\partial x_i}\right)\right\|\quad (i=1,2),\tag{13}$$

which are in a sense sharper than (11), cf. [10]. These relations follow from the facts that $\partial(v-\Pi^1_{\alpha,\pi/2,h}v)/\partial x_i\in V^i_{\alpha,\pi/2,h}$ for $i=1,2$. It is to be noted that, for $\theta\neq\pi/2$, the results still hold if $C_i(\alpha)$ is replaced with $C_i(\alpha,\theta)$ for each of $i=1,2$ and the partial derivative for $i=2$ is done with the directional derivative of v in OB direction.

Thus we can give quantitative interpolation estimates if we succeed in evaluating or bounding the constants $C_i(\alpha,\theta)$'s explicitly. So we will give upper bounds of these constants as fairly simple functions of α and θ. Notice here that each of such constants can be characterized by minimization of a kind of Rayleigh quotient. Then it is equivalent to finding the minimum eigenvalue of a certain eigenvalue problem expressed by a weak formulation, which is further expressed by a partial differential equation with some auxiliary conditions.

For later purposes, let us explain the cases of $C_0(\alpha,\theta)$ and $C_1(\alpha,\theta)$ as examples. From (8), $C_0(\alpha,\theta)$ is characterized by using a kind of Rayleigh quotient:

$$C_0(\alpha,\theta)^{-2}=\inf_{v\in V^0_{\alpha,\theta}\setminus\{0\}}\frac{\|Dv\|^2}{\|v\|^2},\tag{14}$$

where all notations and quantities are for $T_{\alpha,\theta}$. The infimum in the right-hand side is actually a minimum, and it is the smallest eigenvalue of the eigenvalue problem: *Find* $\lambda\in\mathbf{R}$ *and* $u\in V^0_{\alpha,\theta}\setminus\{0\}$ *that satisfy*

$$(\nabla u,\nabla v)_{T_{\alpha,\theta}}=\lambda(u,v)_{T_{\alpha,\theta}}\quad (\forall v\in V^0_{\alpha,\theta}).\tag{15}$$

Here, $(\cdot,\cdot)_{T_{\alpha,\theta}}$ denotes the inner products of both $L_2(T_{\alpha,\theta})$ and $L_2(T_{\alpha,\theta})^2$, and ∇ is the gradient operator. The present eigenvalue problem is also expressed in terms of a partial differential equation, the linear constraint for $V^0_{\alpha,\theta}$ and the boundary condition [13,14]:

$$-\Delta u=\lambda u \text{ in } T_{\alpha,\theta},\quad \int_{T_{\alpha,\theta}}u(x)\,\mathrm{d}x=0,\quad \frac{\partial u}{\partial n}=0 \text{ on } \partial T_{\alpha,\theta},\tag{16}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on edges, and $\partial T_{\alpha,\theta}$ does the boundary of $T_{\alpha,\theta}$. The above boundary condition is the homogeneous Neumann one, and the desired minimum eigenvalue is also the second (and positive) one for the same problem without the linear constraint.

For $C_1(\alpha,\theta)$, it is characterized in essentially the same fashion as (14) and (15), but the associated space $V^0_{\alpha,\theta}$ must be replaced with $V^1_{\alpha,\theta}$. On the other hand, the equations corresponding to (16) become more complicated [13,14]:

$$-\Delta u=\lambda u \text{ in } T_{\alpha,\theta},\quad \int_0^1 u(x_1,0)\,\mathrm{d}x_1=0,$$
$$\frac{\partial u}{\partial n}=\begin{cases}0 & \text{on edges } OB \text{ and } AB,\\ c & \text{on edge } OA,\end{cases}\tag{17}$$

where c denotes an unknown constant to be decided with u and λ.

The other constants are characterized similarly, but the partial differential equations related to $C_3(\alpha,\theta)$ and $C_4(\alpha,\theta)$ are of fourth order and are more difficult to deal with than the second-order equations like above, cf. [2,5]. Since $T_{\alpha,\theta}$ is a triangle, it is in general difficult to solve such eigenvalue problems explicitly. However, in certain special cases, we can achieve such aims as we will see later.

3. Dependence of constants on θ

This section is devoted to analysis of the effects of the maximum interior angle θ on $C_i(\alpha,\theta)$'s for fixed α. For $C_3(\alpha,\theta)$, the well-known maximum angle condition was derived in [3]. However, the results reported there are not fully quantitative, so that we give here more quantitative estimates for the constants including $C_3(\alpha,\theta)$.

To this end, let us introduce the following simple affine transformation between $x=\{x_1,x_2\}\in T_{\alpha,\theta}$ and $\xi=\{\xi_1,\xi_2\}\in T_\alpha$:

$$\xi_1=x_1-x_2/\tan\theta,\quad \xi_2=x_2/\sin\theta.\tag{18}$$

This transformation is a bit different from that in [3]. By eigenvalue analysis of matrices resulting from the above transformation in the Rayleigh quotients like (14), we obtain the following results.

Theorem 1. *For* $h=1$, *it holds for each* $\alpha\in]0,1]$ *that*

$$C_i(\alpha,\theta)\leqslant\phi_i(\theta)C_i(\alpha)\quad \left(0\leqslant i\leqslant 4;\frac{\pi}{3}\leqslant\cos^{-1}\frac{\alpha}{2}\leqslant\theta<\pi\right),\tag{19}$$

where

$$\phi_i(\theta)=\sqrt{1+|\cos\theta|}\quad (i=0,1,2),$$
$$\phi_3(\theta)=\frac{1+|\cos\theta|}{\sqrt{1-|\cos\theta|}},\quad \phi_4(\theta)=1+|\cos\theta|.\tag{20}$$

Remark. The function form for $\phi_3(\theta)$ is consistent with the maximum angle condition in [3], since $\phi_3(\theta)$ is bounded on $[\pi/3, \pi-\delta]$ for each sufficiently small $\delta>0$. Notice also that $C_3(\alpha) \leqslant C_1 = C_2$ for $\alpha \leqslant 1$, as will be shown in the subsequent section. The other ϕ_i's are uniformly bounded on $[\pi/3, \pi[$. Moreover, the corresponding result for $C_3(\alpha,\theta)$ by Natterer [15] is expressed in terms of C_3, $\alpha(\leqslant 1)$ and θ as

$$C_3(\alpha,\theta) \leqslant \frac{1+\alpha^2+\sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}}{\sqrt{2\left(1+\alpha^2-\sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}\right)}}C_3. \tag{21}$$

This estimation is, however, not consistent with the maximum angle condition. In fact, the right-hand side of the above diverges to ∞ as $\alpha \downarrow 0$. When $\alpha \approx 1$, our formula for $C_3(\alpha,\theta)$ is numerically comparable to Natterer's, even when $C_3(\alpha)$ in (19) is replaced with C_1. In particular, when $\alpha = 1$, (21) is identical to (19) for $i = 3$.

Proof. We will use the coordinate transformation (18) between $T_{\alpha,\theta}$ and T_α. By simple calculations, we have for $\tilde{v}(\xi_1,\xi_2) = v(x_1,x_2)$ under the present transformation:

$$\sum_{i=1}^{2}\left(\frac{\partial v}{\partial x_i}\right)^2 = \frac{1}{\sin^2\theta}\left[\left(\frac{\partial \tilde{v}}{\partial \xi_1}\right)^2 - 2\cos\theta\frac{\partial \tilde{v}}{\partial \xi_1}\frac{\partial \tilde{v}}{\partial \xi_2} + \left(\frac{\partial \tilde{v}}{\partial \xi_2}\right)^2\right],$$

where v and $\tilde{v}$ are assumed to be sufficiently smooth. Then we can easily derive

$$\frac{1-|\cos\theta|}{\sin^2\theta}\sum_{i=1}^{2}\left(\frac{\partial \tilde{v}}{\partial \xi_i}\right)^2 \leqslant \sum_{i=1}^{2}\left(\frac{\partial v}{\partial x_i}\right)^2 \leqslant \frac{1+|\cos\theta|}{\sin^2\theta}\sum_{i=1}^{2}\left(\frac{\partial \tilde{v}}{\partial \xi_i}\right)^2.$$

Moreover, the Jacobian of the present transformation is evaluated as $\partial(x_1,x_2)/\partial(\xi_1,\xi_2) = \sin\theta$. From these estimates, we have

$$\|v\|^2_{T_{\alpha,\theta}} = \sin\theta\|\tilde{v}\|^2_{T_\alpha},$$
$$\frac{1-|\cos\theta|}{\sin\theta}\|D\tilde{v}\|^2_{T_\alpha} \leqslant \|Dv\|^2_{T_{\alpha,\theta}} \leqslant \frac{1+|\cos\theta|}{\sin\theta}\|D\tilde{v}\|^2_{T_\alpha}, \tag{a}$$

where $\|\cdot\|_{T_{\alpha,\theta}}$, for example, denotes $\|\cdot\|$ for $T_{\alpha,\theta}$. The results for $i = 0, 1, 2$ are now easy to obtain by using the above and the definitions of the constants $C_i(\alpha,\theta)$'s.

Similarly, we obtain

$$\sum_{i,j=1}^{2}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 = \frac{1}{\sin^4\theta}\Bigg[\left(\frac{\partial^2 \tilde{v}}{\partial \xi_1^2}\right)^2 + \left(\frac{\partial^2 \tilde{v}}{\partial \xi_2^2}\right)^2 + 2(1+\cos^2\theta)\left(\frac{\partial^2 \tilde{v}}{\partial \xi_1 \partial \xi_2}\right)^2 + 2\cos^2\theta\frac{\partial^2 \tilde{v}}{\partial \xi_1^2}\frac{\partial^2 \tilde{v}}{\partial \xi_2^2} - 4\cos\theta\frac{\partial^2 \tilde{v}}{\partial \xi_1^2}\frac{\partial^2 \tilde{v}}{\partial \xi_1 \partial \xi_2} - 4\cos\theta\frac{\partial^2 \tilde{v}}{\partial \xi_2^2}\frac{\partial^2 \tilde{v}}{\partial \xi_1 \partial \xi_2}\Bigg].$$

Let us consider the following real symmetric matrix related to the quadratic form in the right-hand side of the above expression:

$$\frac{1}{\sin^4\theta}\begin{pmatrix} 1 & \cos^2\theta & -\sqrt{2}\cos\theta \\ \cos^2\theta & 1 & -\sqrt{2}\cos\theta \\ -\sqrt{2}\cos\theta & -\sqrt{2}\cos\theta & 1+\cos^2\theta \end{pmatrix}.$$

We can see that this has three eigenvalues $1/(1+|\cos\theta|)^2$, $1/(1-|\cos\theta|^2)$ and $1/(1-|\cos\theta|)^2$, so that we have the estimates

$$\frac{1}{(1+|\cos\theta|)^2}\sum_{i,j=1}^{2}\left(\frac{\partial^2 \tilde{v}}{\partial \xi_i \partial \xi_j}\right)^2 \leqslant \sum_{i,j=1}^{2}\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 \leqslant \frac{1}{(1-|\cos\theta|)^2}\sum_{i,j=1}^{2}\left(\frac{\partial^2 \tilde{v}}{\partial \xi_i \partial \xi_j}\right)^2.$$

As (a), we have now

$$\frac{\sin\theta}{(1+|\cos\theta|)^2}\|D^2\tilde{v}\|^2_{T_\alpha} \leqslant \|D^2 v\|^2_{T_{\alpha,\theta}} \leqslant \frac{\sin\theta}{(1-|\cos\theta|)^2}\|D^2\tilde{v}\|^2_{T_\alpha}. \tag{b}$$

Applying (a) and (b) to the definitions of the constants, we have the results for $i = 3, 4$. □

4. Dependence of constants on α

Up to now, we have given some basic results for dependence of error constants on h and θ. In this section, we will consider the dependence of such constants on α when $\theta = \pi/2$. With this regard, we owe much the following results to the analysis by Babuška and Aziz [3]. In particular, the estimation $C_3(\alpha) \leqslant C_1$ below is an important consequence derived in [3] and also in [13,18], and so we here call C_1 the Babuška–Aziz constant.

Theorem 2. *For $h = 1$ and $\theta = \pi/2$, $C_i(\alpha)$ $(0 \leqslant i \leqslant 4)$ are continuous positive-valued functions of $\alpha \in]0,+\infty[$ (here we consider also for $\alpha > 1$). In addition, except for $i = 3$, they are monotonically increasing in α. Thus,*

$$C_i(\alpha) \leqslant C_i; \quad \forall \alpha \in]0,1] \ (i = 0,1,2,4). \tag{22}$$

On the other hand, it holds for $i = 3$ and $\alpha \in]0,1]$ that

$$C_3(\alpha) \leqslant \max\{C_1(\alpha), C_2(\alpha)\} \leqslant C_1(= C_2). \tag{23}$$

Proof. We just give sketches since the arguments employed here are standard. It is convenient to consider over the common domain T by applying a simple coordinate transformation in [3] to T_α. For the continuity, we first show the uniform boundedness over compact intervals, which assures the existence of $\overline{\lim}_{\beta\to\alpha}C_i(\beta)$ and $\underline{\lim}_{\beta\to\alpha}C_i(\beta)$ for each $\alpha > 0$. Then we can prove the continuity by adopting the weakly lower semi-continuity of L_2-norm and the Rellich compactness theorem. The monotonicity and (23) can be concluded completely as in [3]. □

Consequently, we can give P_0 and P_1 interpolation error estimates in terms of C_0, C_1 and C_4. That is, from the preceding considerations, we have $C_0(\alpha,\theta,h) \leqslant C_0\phi_0(\theta)h$, $C_3(\alpha,\theta,h) \leqslant C_1\phi_3(\theta)h$ and $C_4(\alpha,\theta,h) \leqslant C_4\phi_4(\theta)h^2$, so that (10) through (12) become

$$\|v - \Pi^0_{\alpha,\theta,h}v\| \leqslant C_0\phi_0(\theta)h\|Dv\|; \quad \forall v \in H^1(T_{\alpha,\theta,h}), \tag{24}$$

$$\|D(v - \Pi^1_{\alpha,\theta,h}v)\| \leqslant C_1\phi_3(\theta)h\|D^2v\|; \quad \forall v \in H^2(T_{\alpha,\theta,h}), \tag{25}$$

$$\|v - \Pi^1_{\alpha,\theta,h}v\| \leqslant C_4\phi_4(\theta)h^2\|D^2v\|; \quad \forall v \in H^2(T_{\alpha,\theta,h}). \tag{26}$$

These may be rough but are still correct upper bounds. As was already noted, such error bounds are available for triangles of general configuration by applying appropriate congruent transformations [3,7,8,10].

Thus we can obtain a quantitative error bound for P_0 interpolation of $v \in H^1(T_{\alpha,\theta,h})$ and those for P_1 interpolation of $v \in H^2(T_{\alpha,\theta,h})$, provided that numerical values or concrete upper bounds of $C_0, C_1 = C_2$ and C_4 are known. Rough upper bounds of these constants can be given even by manual calculation [5,15]. For example, we found that $C_4 \leqslant \sqrt{12}$ (see Acknowledgements of this paper). To obtain accurate upper (and lower) bounds, however, we need numerical computations with verification. Quite fortunately, we can get exact values for C_0 and C_1 (Babuška–Aziz constant) as will be shown in the subsequent section. An upper bound for C_3 was first given by Natterer [15]. By numerical computations without verification, it is now known that $C_3 \approx 0.489$ [2,11,18]. The relation between C_3 and C_1 was fully discussed in [18,13], and in certain cases C_1 is more essential than C_3 itself as we already noted, cf. [10]. We should also mention that C_1 was verified numerically in [13,14] with estimate $0.492 \leqslant C_1 \leqslant 0.494$. Thus 0.493 or so is a nice upper bound to C_3 for most of practical purposes. In fact, 0.5 is recommended in [18] for use as an upper bound for C_3.

5. Determination of C_0 and C_1

First let us determine C_0 exactly. Actually, its exact value is already known, see e.g. [13,14]. However, we here state the results with a proof, since the underlying idea is somewhat common to the more complicated case of C_1.

Theorem 3. *With regard to C_0, i.e., $C_0(\alpha,\theta)$ for $\alpha = 1$ and $\theta = \pi/2$, it holds that $C_0 = 1/\pi$.*

Proof. We will prove in two steps, each of which is based on rather well-known arguments and techniques. The triangular domain to be considered here is T.

(1) Let Ω be a unit square domain: $\Omega = \{x = \{x_1, x_2\} \in \mathbf{R}^2; 0 < x_1, x_2 < 1\}$. Let $\{\lambda, u\} \in \mathbf{R} \times V^0 \setminus \{0\}$ be an arbitrary eigenpair of (15) or (16) for $\{\alpha,\theta\} = \{1, \pi/2\}$, and define the (symmetric) extension $\tilde{u}$ of u to Ω by reflection with respect to the line $x_1 + x_2 = 1$:

$$\tilde{u}(x_1,x_2) = u(x_1,x_2) \quad \text{if } x = \{x_1,x_2\} \in T,$$

$$\tilde{u}(x_1,x_2) = u(1-x_2, 1-x_1) \quad \text{if } x \in \Omega \setminus T.$$

We can find that $\{\lambda, \tilde{u}\}$ is an eigenpair of the eigenvalue problem for Ω:

$$\tilde{u} \in \widetilde{V}^0 \setminus \{0\}; (\nabla\tilde{u}, \nabla\tilde{v})_\Omega = \lambda(\tilde{u},\tilde{v})_\Omega \quad (\forall \tilde{v} \in \widetilde{V}^0), \tag{c}$$

where $(\cdot,\cdot)_\Omega$ denotes the inner products of $L_2(\Omega)$ and $L_2(\Omega)^2$, and $\widetilde{V}^0$ is defined by

$$\widetilde{V}^0 = \left\{\tilde{v} \in H^1(\Omega); \int_\Omega \tilde{v}(x)\,\mathrm{d}x = 0\right\}.$$

Conversely, any eigenpair of (c) with $\tilde{u}$ restricted to T satisfies (15), if $\tilde{u}$ is symmetric with respect to the line $x_1 + x_2 = 1$. Notice here the orthogonal decomposition of $\widetilde{V}^0$ in $H^1(\Omega)$ as well as in $L_2(\Omega)$:

$$\widetilde{V}^0 = \widetilde{V}^0_s \oplus \widetilde{V}^0_a, \begin{cases} \widetilde{V}^0_s = \text{subspace of symmetric functions in } \widetilde{V}^0, \\ \widetilde{V}^0_a = \text{subspace of antisymmetric functions in } \widetilde{V}^0. \end{cases}$$

Consequently, for the present purposes, it suffices to deal with (c) in $\widetilde{V}^0_s$.

(2) As is well known, a complete system of functions for $H^1(\Omega)$ is given by the totality of (orthogonal) eigenfunctions of (c) with $\widetilde{V}^0$ replaced with the whole $H^1(\Omega)$:

$$\phi_{mn}(x_1,x_2) = \cos m\pi x_1 \cos n\pi x_2 \quad (m,n = 0,1,2,3,\ldots).$$

Since we are interested in symmetric eigenfunctions only, we should make a complete system of symmetric functions in $H^1(\Omega)$ from the above: for $m \geqslant n$; $m,n = 0,1,2,3,\ldots$,

$$\psi_{mn}(x_1,x_2) = \phi_{mn}(x_1,x_2) + \phi_{mn}(1-x_2, 1-x_1).$$

These are orthogonal in $L_2(\Omega)$, and also orthogonal with respect to the bilinear form $(\nabla\cdot,\nabla\cdot)_\Omega$ (and in $H^1(\Omega)$). More important to note is that all ψ_{mn}'s for $m \geqslant n$ except for $\psi_{00} \equiv 2$ belong to $\widetilde{V}^0_s$ and are eigenfunctions of (c). Thus the desired eigenvalue λ_0 is π^2 associated to ψ_{10}, and hence $C_0 = 1/\sqrt{\lambda_0} = 1/\pi$. □

Next we determine $C_1 = C_2$. See also [9] for a slightly different approach.

Theorem 4. *The minimum eigenvalue λ_1 associated to $C_1 = C_2$ is equal to the minimum positive solution of the transcendental equation for λ:*

$$\sqrt{\lambda} + \tan\sqrt{\lambda} = 0. \tag{27}$$

The concrete value of λ_1 can be obtained numerically with verification. For example, we find $2.0287 < \sqrt{\lambda_1} < 2.0291$, and hence $C_1 = 1/\sqrt{\lambda_1}$ is bounded as

$$0.49282 < C_1 < 0.49293. \tag{28}$$

Remark. Numerical computation without verification gives $C_1 = 0.49291245\ldots$ The present transcendental equation can be commonly seen in vibration analysis of strings with special boundary conditions [16].

Proof. The use of reflection and trigonometric functions is common to the proof of the preceding theorem.

(1) Let Ω be the same as before. Let $\{\lambda, u\} \in \mathbf{R} \times V^1 \setminus \{0\}$ be an arbitrary eigenpair of (17) for T, and

define the symmetric extension $\tilde{u}$ of u to Ω by reflection. Then $\{\lambda, \tilde{u}\}$ is an eigenpair of the eigenvalue problem for Ω:

$$\tilde{u} \in \widetilde{V}^1 \setminus \{0\}; (\nabla\tilde{u}, \nabla\tilde{v})_\Omega = \lambda(\tilde{u}, \tilde{v})_\Omega \quad (\forall \tilde{v} \in \widetilde{V}^1), \tag{d}$$

where $\widetilde{V}^1$ is defined by

$$\widetilde{V}^1 = \left\{\tilde{v} \in H^1(\Omega); \int_0^1 \tilde{v}(x_1, 0)\,dx_1 = 0,\ \int_0^1 \tilde{v}(1, x_2)\,dx_2 = 0\right\}. \tag{e}$$

Conversely, any eigenpair of (d) with $\tilde{u}$ restricted to T satisfies the weak form of (17), if $\tilde{u}$ is symmetric with respect to the line $x_1 + x_2 = 1$. Notice here the orthogonal decomposition of $\widetilde{V}^1$ in $H^1(\Omega)$ as well as in $L_2(\Omega)$:

$$\widetilde{V}^1 = \widetilde{V}^1_s \oplus \widetilde{V}^1_a,$$

where $\widetilde{V}^1_s$ and $\widetilde{V}^1_a$ are respectively the symmetric and anti-symmetric subspaces of $\widetilde{V}^1$. Consequently, for the present purposes, it suffices to deal with (d) in $\widetilde{V}^1_s$.

(2) We use the complete system of symmetric functions ψ_{mn}'s for $m \geqslant n$; $m, n = 0, 1, 2, 3, \ldots$ in $H^1(\Omega)$ defined in the proof of the preceding theorem. From (e), the condition for a symmetric $\tilde{v} \in H^1(\Omega)$ to belong to $\widetilde{V}^1_s$ is expressed by

$$2a_{00} + \sum_{m=1}^{\infty}(-1)^m a_{m0} = 0 \quad \text{for } \tilde{v} = \sum_{m \geqslant n \geqslant 0}^{\infty} a_{mn}\psi_{mn}$$

$$\text{with} \sum_{m \geqslant n \geqslant 0}^{\infty}(1 + m^2 + n^2)a_{mn}^2 < +\infty,$$

where a_{mn}'s are real coefficients, and we can show the series $\sum_{m=1}^{\infty}(-1)^m a_{m0}$ is absolutely convergent under the conditions imposed on the coefficients. Eliminating a_{00} by the above equation, $\forall \tilde{v} \in \widetilde{V}^1_s$ is expressed by

$$\tilde{v} = \sum_{m=1}^{\infty} a_{m0}[\psi_{m0} - (-1)^m] + \sum_{m \geqslant n \geqslant 1}^{\infty} a_{mn}\psi_{mn}. \tag{f}$$

Clearly, ψ_{mn}'s for $m \geqslant n \geqslant 1$ are eigenfunctions of (d) with the homogeneous Neumann boundary condition, and the minimum of the associated eigenvalues is $2\pi^2$.

(3) Taking notice of (*f*), $\widetilde{V}^1_s$ is expressed by the direct sum

$$\widetilde{V}^1_s = W_1 \oplus W_2,$$

where $W_1 =$ closure of linear combinations of $\psi_{m0} - (-1)^m$ $(m = 1, 2, 3, \ldots)$ and $W_2 =$ closure of linear combinations of ψ_{mn} $(m \geqslant n \geqslant 1)$. Here, W_1 and W_2 are orthogonal to each other in both $L_2(\Omega)$ and $H^1(\Omega)$, and moreover, from the observation in step (2), all the eigenfunctions in W_2 are known. Consequently, our aim will be attained if we obtain the minimum of eigenvalues associated with eigenfunctions in W_1. If it is smaller than $2\pi^2$, the obtained one is nothing but the desired eigenvalue λ_1.

(4) Let us now solve the eigenvalue problem (d) in W_1 by expressing $\tilde{u} \in W_1 \setminus \{0\}$ as

$$\tilde{u} = \sum_{m=1}^{\infty} a_m\varphi_m \quad \text{with} \sum_{m=1}^{\infty} m^2 a_m^2 < +\infty, \tag{g}$$

where $\varphi_m(x_1, x_2) = \psi_{m0}(x_1, x_2) - (-1)^m = \cos m\pi x_1 + \cos m\pi(1 - x_2) - (-1)^m$ $(m \in \mathbf{N})$. Thus, using the theory of Fourier series with the inequality in (*g*) taken into account, $\tilde{u} \in W_1$ must be of the form, for an unknown single-variable function $g = g(t)$,

$$\tilde{u}(x_1, x_2) = g(x_1) + g(1 - x_2).$$

Substituting the above into (d), we have

$$-g''(t) = \lambda g(t)(0 < t < 1), \quad g'(0) = 0, \quad g(1) + \int_0^1 g(t)\,dt = 0.$$

Notice in this derivation that $\tilde{v}$ in (d) can be taken from whole $\widetilde{V}^1$ so that (17) is available, since W_1 is orthogonal to W_2 and $\widetilde{V}^1_a$ both in $L_2(\Omega)$ and $H^1(\Omega)$. Solving this eigenvalue problem, we obtain (27). Clearly, the minimum positive solution of (27) lies in the interval $]\pi^2/4, \pi^2[$, and is the unique solution there. It is surely smaller than $2\pi^2$, and is exactly the desired eigenvalue λ_1. Moreover, an eigenfunction associated to λ_1 is $\tilde{u}(x_1, x_2) = \cos\sqrt{\lambda_1}x_1 + \cos\sqrt{\lambda_1}(1 - x_2)$.

(5) To obtain $\sqrt{\lambda_1} \in]\pi/2, \pi[$ numerically with verification, we can use various methods. Here we just use a method based on modification of the equation $t + \tan t = 0$ for $t > 0$: Let us find the minimum positive zero of

$$f(t) := \frac{\cos t}{2} + \frac{\sin t}{2t} = \sum_{m=0}^{\infty}\frac{(-1)^m(m+1)t^{2m}}{(2m+1)!} \quad (t > 0).$$

The series appearing above is an alternating one, and the absolute value of each term for fixed t converges to 0 as $m \to \infty$, monotonically for sufficiently large m. Moreover, $f(t)$ is monotonically decreasing for $0 < t < \pi$. Thus, as is well known in elementary calculus, we can compute upper and lower bounds for the minimum zero t_0 by utilizing appropriate partial sums: $f_n(t) :=$ partial sum up to the term of $m = n$. It is to be noted here that, at least in principle, all the computations can be performed in the finite-digit binary arithmetic without computer errors, provided that t is a rational number. For example, by taking $n = 4$, 5, we can bound $t_0 = \sqrt{\lambda_1}$ as $2.0287 < t_0 < 2.0291$, since $f(2.0291) < f_4(2.0291) < 0$ (even n) and $f(2.0287) > f_5(2.0287) > 0$ (odd n). □

6. Asymptotic behaviors of constants as $\alpha \to +0$

Moreover, we can analyze the asymptotic behaviors of the constants $C_i(\alpha)$'s as $\alpha \to +0$, cf. [12]. In particular, the right limit values $C_i(+0)$'s are given by zeros of certain transcendental equations (derived from eigenvalue problems of ordinary differential equations, ODE's) in terms of the hypergeometric functions [20]. For example, $C_2(+0)^{-1}$ is equal to the first positive zero of the Bessel function $J_0(z)$.

For the analysis, we use various techniques including compactness arguments. We will publish the detailed analyses and results elsewhere, since they become rather

Table 1
Right limits of $C_i(\alpha)$: $C_i(+0) = \lim_{\alpha\to+0} C_i(\alpha) = 1/\sqrt{\lambda^{(i)}}$ $(0 \leqslant i \leqslant 4)$

i	ODE's for eigenvalue problems ($x \in [0,1]$)	Constraints and/or boundary conditions	Numerical values for $C_i(+0)$'s
0	$((1-x)u'(x))' = \lambda^{(0)}(1-x)u(x)$	$\int_0^1 (1-x)u(x)\mathrm{d}x = u'(0) = 0$	0.26098
1	$((1-x)u'(x))' = \lambda^{(1)}(1-x)u(x) + C$ (C: *unknown constant*)	$\int_0^1 u(x)\mathrm{d}x = u'(0) = 0$	0.32454
2	$((1-x)u'(x))' = \lambda^{(2)}(1-x)u(x)$	$u(0) = 0$	0.41583
3	$((1-x)u''(x))'' = \lambda^{(3)}((1-x)u'(x))'$ (*reduces to case: i = 1*)	$u(0) = u(1) = u''(0) = 0$	0.32454
4	$((1-x)u''(x))'' = \lambda^{(4)}(1-x)u(x)$	$u(0) = u(1) = u''(0) = 0$	0.10790

lengthy. Instead, we list up the related ODE's with the constraint and/or boundary conditions in Table 1.

7. Nonconforming P_1 triangle

We have mainly considered the conforming P_1 triangle, which can naturally construct subspaces of H^1 space over the entire domain. But there also exists a non-conforming counterpart, which is also based on $\mathscr{P}_1$ but uses as nodes the midpoints of edges or edges themselves [19]. Analysis of such an element is more complicated, since we must additionally evaluate the errors induced by the interelement discontinuity of the approximate functions. Still we can obtain some results for the interpolation errors. The estimates shown below are based on the preceding results for the usual P_0 and P_1 interpolations. They may be fairly rough, but can be used for some purposes. To give sharper estimates, we must introduce and analyze some new constants.

We define here the non-conforming P_1 interpolation operator $\Pi^{1,n}_{\alpha,\theta,h}$ as follows: for $\forall v \in H^1(T_{\alpha,\theta,h})$, $\Pi^{1,n}_{\alpha,\theta,h}v$ is a function in $\mathscr{P}_1$ such that

$$\int_e \Pi^{1,n}_{\alpha,\theta,h} v\,\mathrm{d}s = \int_e v\,\mathrm{d}s \quad \text{for edges } e \quad (e = OA, AB, OB \text{ of } T_{\alpha,\theta,h}), \tag{29}$$

where $\mathrm{d}s$ denotes the infinitesimal line element on edges.

Then we have the following results for $v \in H^2(T_{\alpha,\theta,h})$ in terms of the constants introduced for the original P_0 and P_1 interpolations:

$$\left\| \frac{\partial(v - \Pi^{1,n}_{\alpha,\theta,h}v)}{\partial x_i} \right\| \leqslant C_0(\alpha,\theta) h \left\| D\left(\frac{\partial v}{\partial x_i}\right) \right\| \quad (i = 1,2), \tag{30}$$

$$\|v - \Pi^{1,n}_{\alpha,\theta,h}v\| \leqslant C_0(\alpha,\theta) \min\{C_1(\alpha,\theta), C_2(\alpha,\theta)\} h^2 \|D^2 v\|. \tag{31}$$

To show (30), we use (29) and the Gauss formula to derive $\int_{T_{\alpha,\theta,h}} \partial(v - \Pi^{1,n}_{\alpha,\theta,h}v)/\partial x_i \,\mathrm{d}x = 0$ for $i = 1,2$. Then we can easily obtain (30) by noting the definition of $C_0(\alpha,\theta)$. To derive (31), we should evaluate $\|v - \Pi^{1,n}_{\alpha,\theta,h}v\| / \|D(v - \Pi^{1,n}_{\alpha,\theta,h}v)\|$ and $\|D(v - \Pi^{1,n}_{\alpha,\theta,h}v)\| / \|D^2 v\|$. The former can be evaluated by using $C_1(\alpha,\theta)$ and $C_2(\alpha,\theta)$, while the latter can be done by (30).

8. Numerical results

We performed numerical computations to see the actual dependence of various constants on α and θ. Here, we just show the results for $C_1(\alpha)$, $C_2(\alpha)$ and $C_3(\alpha)$ by the P_1 FEM with the uniform triangulation of the domain T_α. In such calculations, T_α is subdivided into a number of small congruent triangles $T_{\alpha,\pi/2,h}$ with $h = 1/20$. The penalty method in [18] was also adopted to calculate $C_3(\alpha)$ approximately. The resulting approximate problems are matrix eigenvalue ones, and can be solved numerically if the linear constraint conditions imposed on eigenfunctions are appropriately dealt with.

Fig. 2 illustrates the graphs of approximate $C_i(\alpha)$'s $(i = 1,2,3)$ versus $\alpha \in]0,1]$. The exact value $C_1 = C_2$ at $\alpha = 1$ is also included as a horizontal line. At $\alpha = 1$, the approximate values coincide well with the exact one, and, for general α, the monotonically increasing behaviors of these functions are also well represented. The present numerical results suggest that $C_3(\alpha)$ is also monotonically increasing, but we have not succeeded in proving such a conjecture. Moreover, when $\alpha \approx 0$, the numerical results agree well with the exact limits given in Table 1 based on the asymptotic analysis.

As a simple example of application of our results, let us consider a kind of a posteriori estimate for approximation of $\lambda_0 = C_0^{-2}$ by P_1 FEM. By Schultz [17] and many others, we have the following a priori error estimate for the approximation λ_{h0} to λ_0:

$$\lambda_0 \leqslant \lambda_{h0} \leqslant \lambda_0 + \frac{(\widetilde{C}_3 \tilde{h} \lambda_0)^2}{(1 - \widetilde{C}_3^2 \tilde{h}^2 \lambda_0)^2} =: \varphi(\lambda_0) \quad (\widetilde{C}_3^2 \tilde{h}^2 \lambda_0 < 1), \tag{32}$$

where $\widetilde{C}_3$ is a positive constant such that $\widetilde{C}_3 \geqslant C_3(\alpha,\theta)$ for all $T_{\alpha,\theta,h}$ in the triangulation, and $\tilde{h} = \max h$ in the triangulation. Since the function $\varphi(\lambda_0)$ above is monotonically increasing, it has the inverse function. Thus we have the following a posteriori estimate for λ_{h0}:

$$\varphi^{-1}(\lambda_{h0}) \leqslant \lambda_0 \leqslant \lambda_{h0}. \tag{33}$$

Table 2 gives an application of (33) based on numerical results by the P_1 FEM. Here, the employed meshes are uniform ones composed of small triangles similar to the entire domain T. The values of parameters $\widetilde{C}_3$ and $\tilde{h}$ that are necessary to use (32) and (33) are also shown in the

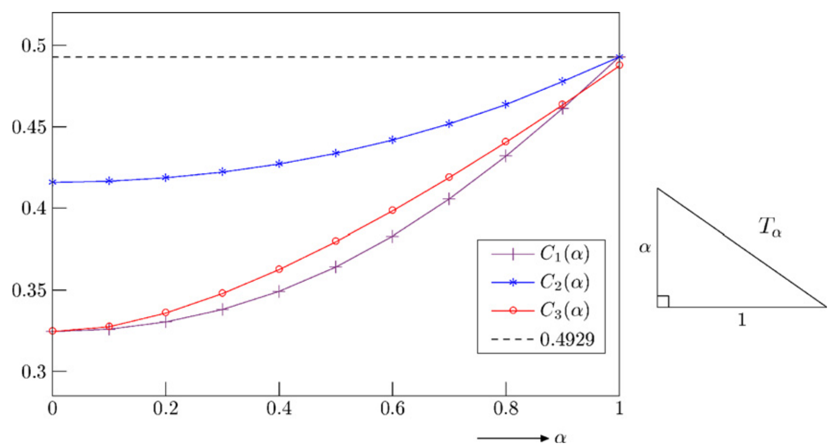


Fig. 2. Numerical results for $C_1(\alpha)$, $C_2(\alpha)$ and $C_3(\alpha)$ $(0 < \alpha \leqslant 1)$.

Table 2
A posteriori estimates for C_0

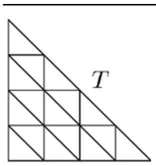
	N	Bounds for λ_0	Bounds for C_0
	2	$5.9890 < \lambda_0 < 11.7154$	$0.2921 < C_0 < 0.4086$
	3	$7.8535 < \lambda_0 < 10.6563$	$0.3063 < C_0 < 0.3568$
	4	$8.7222 < \lambda_0 < 10.3156$	$0.3113 < C_0 < 0.3386$
	8	$9.5982 < \lambda_0 < 9.9867$	$0.3164 < C_0 < 0.3278$
	16	$9.8042 < \lambda_0 < 9.9000$	$0.31782 < C_0 < 0.31937$
	32	$9.8535 < \lambda_0 < 9.8774$	$0.31818 < C_0 < 0.31856$
$\tilde{h} = 1/N$	64	$9.8656 < \lambda_0 < 9.8716$	$0.31827 < C_0 < 0.31838$
$N = 4$ above $\tilde{C}_3 = 0.5$	∞	$\lambda_0 = \pi^2 = 9.8696\ldots$	$C_0 = 1/\pi = 0.318309\ldots$

table. We can observe that this simple method can actually bound C_0 from both above and below. It is straightforward to apply it to give a posteriori estimates to general $C_0(\alpha, \theta)$. By slight modification, it can be also used to bound $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$.

9. Concluding remarks

We have obtained some explicit relations for the dependence of a few interpolation error constants on geometric parameters of triangular finite elements. In particular, we have succeeded in determining the Babuška–Aziz constant from a very simple equation. We can effectively utilize these results to give upper bounds of the a priori and a posteriori error estimates of finite element solutions based on the P_1 and/or P_0 approximate functions. To obtain more clear picture for the dependence of the interpolation error constants, we should also perform various analyses including numerical analysis with verifications, asymptotic analysis etc. We will continue such study, and more detailed results will be reported in due course.

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